

# R-mode oscillations and rocket term in rotating superfluid neutron stars

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We discuss a novel damping mechanism of r-mode oscillations in neutron stars due to processes that change the number of protons, neutrons and electrons. Deviations from equilibrium between the number densities of the various species lead to the appearance in the Euler equations of the system of a dissipative force, the so-called rocket term. Such a force affects the evolution of the r-mode oscillations of a rotating neutron star and we estimate the corresponding damping timescale. In the description of the system we employ a two-fluid model, with one fluid consisting of all the charged components locked together by the electromagnetic interaction, while the second fluid consists of superfluid neutrons. Both components can oscillate and we consider two different kind of r-mode oscillations, one predominantly associated with comoving displacements of the two fluids and a second one associated with countermoving, out of phase, displacements. In the former case we find that the dissipation mechanism associated with the rocket term is not effective in preventing the growth of r-mode instabilities. In the latter case the rocket term prevents the growth of the r-mode instability for temperatures larger than about  $10^9$  K. In our analysis we include the mutual friction dissipative process between the neutron superfluid and the charged component. In order to simplify the computation we neglect the interaction between the two r-mode oscillations as well as effects related with the crust of the star. Moreover, we use a simplified model of neutron star assuming a uniform mass distribution.

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## I. INTRODUCTION

Rapidly rotating neutron stars have been the subject of intensive investigations in the last years. Of particular interest are neutron star oscillations, which might be useful to shed light on the internal structure of these stars [1, 2]. There are various modes of oscillations, among them r-mode oscillations are probably the most interesting ones, because they provide a severe limitation on the star's rotation frequency through coupling to gravitational radiation emission [3, 4]. These oscillations can be damped by various dissipative process that can take place in the star [5, 6]. However, if dissipative phenomena are not strong enough, these oscillations will grow exponentially fast in time until the star slows down to a rotation frequency where some dissipative mechanism sets in. Since neutron stars are observed to rotate at very high frequencies, any model of neutron star must provide an efficient mechanism of dissipation of r-mode oscillations. In this way, the study of r-mode damping is useful in constraining the stellar structure and can be used to rule out some exotic phases of matter [7].

Standard neutron stars are stellar objects with a mass of about  $1.4M_\odot$  and a radius of about 10 Km. They are believed to have a crust of about 1 Km, with an outer part made of a lattice of ions embedded in a liquid of electrons and an inner part made of nuclei embedded in a liquid of  $^1S_0$  superfluid neutrons. In the interior of the star nuclei are melted and both neutrons and protons are expected to condense into BCS-like superfluids. However, neutron interaction in the  $^1S_0$  state at supernuclear matter density is repulsive, but it is still possible to form Cooper pairs in the  $^3P_2$  channel [8]. The proton density is much smaller than the neutron density and they can form  $pp$  Cooper pairs in the isotropic  $^1S_0$  channel. Pairing between protons and neutrons does not take place because of the large mismatch between their Fermi energies. In the core of neutron stars muons might be present (when  $\mu_e > m_\mu$ ), or deconfined quarks in a color superconducting phase, moreover pion or kaon condensates might be realized. In the present paper we shall not consider any of these possibilities and assume the core of the neutron star comprises only neutrons, protons and electrons.

R-mode oscillations have been studied extensively in the literature and various damping mechanisms have been proposed [1]. Shear and bulk viscosity are able to suppress the instability in two different ranges of temperatures [5, 6]. For temperatures smaller than about  $10^5$  K shear viscosity will always suppress the instability, but with increasing temperature the effect of shear viscosity is gradually suppressed. For very high temperatures, bulk viscosity becomes an efficient mechanism for damping r-mode oscillations. In the superfluid phase bulk viscosity is suppressed by Pauli

blocking, but at temperatures above than  $10^{10}$  K -  $10^{11}$  K standard nuclear matter is believed to be in the normal phase and for temperatures larger than  $10^{10}$  K bulk viscosity is able to damp the instability. Therefore, one has an “instability window” for standard neutron stars corresponding to a range of temperature approximately given by  $10^6$  K -  $10^{10}$  K. The exact values depend on the details of the model considered. The instability window is in part reduced by the “surface rubbing” between the core and the crust of the star [9, 10]. This mechanism results in a viscous boundary layer between the core and the crust of the star which damps r-mode oscillations for temperatures less than about  $10^{10}$  K and for sufficiently small frequencies.

In Refs. [11–14] it is studied the effect of mutual friction in reducing the instability window. It is shown that the typical timescale of mutual friction is of the order of  $10^4$  s and is therefore too long for damping the r-mode instability. Indeed, the timescale associated with gravitational wave emission is of the order of few seconds (for a millisecond pulsar). However, mutual friction can reduce the instability for certain values of the entrainment parameter [12] or for large values of the drag parameter [14].

In the present paper we study a novel dissipative processes associated with the change in the number of protons, neutrons and electrons. In real neutron stars these processes can take place in the outer core and in the inner crust of the star and are related to beta decays and interactions between the neutron fluid and the crust. Both these processes lead to the appearance of a dissipative force (the so-called rocket term) in the Euler equations of the system. This force is due to the fact that when two or more fluids move with different velocities a change of one component into the other results in a momentum transfer between the fluids. This change in momentum is not reversible, because it is always the faster moving fluid that will lose momentum. The resulting dissipative force is proportional to the mass rate change, and to the relative velocity between the fluids. The name “rocket term” reminds that the same phenomenon takes place in the dynamical evolution of a rocket whose mass is changing in time as it consumes its fuel. As far as we know, the dissipative force due to the rocket term has not been considered in the context of r-mode oscillations. In the mass conservation laws, it is in general assumed that the neutron, proton and electron numbers are separately conserved quantities.

In order to simplify the analysis we consider a simplified model of neutron star consisting of a fluid of neutrons, protons and electrons and no crust. Protons and electrons are locked together by the electromagnetic interaction and therefore we consider that the system consists of two fluids. As a further simplification we assume that the star has a uniform mass distribution with density  $\rho = 2.5\rho_0$  and a radius of 10 km. Since this simplified model of star does not comprise a crust we consider only number changing processes associated with weak interactions.

In our analysis we consider two different r-mode oscillations. One is associated predominantly with toroidal comoving displacements and the second one is dominated by toroidal countermoving displacements. We shall refer to these oscillations as “standard” r-modes and as “superfluid” r-modes, respectively. These two modes decouple for a star made by uniform and incompressible matter, and we shall restrict to treat such a case. We find that the dissipative mechanism associated with the rocket term is not efficient in damping standard r-mode oscillations. On the other hand, this mechanism becomes efficient in damping the superfluid r-mode instability at temperatures of the order of  $10^9$  K. Therefore this mechanism becomes efficient before bulk viscosity starts to be able to damp r-mode oscillations. In agreement with [13] we find that the superfluid r-mode oscillations are damped as well by the mutual friction force for sufficiently large values of the entrainment parameter.

This paper is organized as follows. In Section II we review the hydrodynamic equations in the two fluid approximation. In the Euler equations we neglect shear and bulk viscosities but consider the effects due to the mutual friction force and to the rocket term force. In Section III we present the differential equations governing the deviations from equilibrium of a rotating two fluid system. We study the linearized problem, neglecting deformations of the star due to rotation and assuming that the star has a uniform density. Although not very realistic, these assumptions allow us to simplify the study of the r-modes, while concentrating on the strength of the effect of the rocket term. The contribution of Urca processes to particle mass creation rates are evaluated in Section IV. In Section V we evaluate the timescale corresponding to the rocket term force and compare the result with the timescale associated with other dissipative processes. We draw our conclusions in Section VI. In Appendix A we report some details about the evolution equations for the comoving and countermoving displacements. The expressions of the decay widths of Urca processes in presence of various  $nn$  condensates are reported in Appendix B.

## II. HYDRODYNAMICAL EQUATIONS FOR THE MULTIFLUID NEUTRON STAR MODEL

The equations describing the dissipative processes of neutrons, protons and electrons in the outer core of a standard neutron star have been studied in detail in Ref. [11]. In general, the entropy production rate depends on 19 independent coefficients which are related to the various dissipative processes. However, for vanishing temperature and neglecting viscosities one obtains the expression given in Ref. [15] which depends only on two different coefficients. One is related to mutual friction and the second one with the so-called rocket term. Here we review the basic

hydrodynamical equations in presence of these two different dissipative mechanisms, neglecting the presence of shear and bulk viscosities.

For a system consisting of neutrons, protons and electrons, the mass conservation law is given by, see *e.g.* [11, 15],

$$\partial_t \rho_x + \nabla_i (\rho_x v_x^i) = \Gamma_x, \quad (1)$$

where  $\Gamma_x$  is the particle mass creation rate per unit volume and the index  $x = n, p, e$  refer to the particle species, that is, neutrons, protons and electrons. In these equations we have considered that some process can convert neutrons in protons and electrons and *vice versa*. Therefore, we are assuming that the various components are not separately conserved. One possible mechanism leading to a change in the particle number densities is given by the weak processes

$$n \rightarrow p + e^- + \bar{\nu}_e, \quad p + e^- \rightarrow n + \nu_e. \quad (2)$$

These reactions lead to a change in the chemical potentials of the various species and therefore are associated with number density changes.

A different process is the so-called “transfusion”, see Ref. [16]. In this process when a compression takes place, the ionic constituent of the crust are squeezed and part of their neutron content is released and augments the superfluid neutron component. The opposite mechanism, related to a reduction of the pressure leads to the neutron capture by the ions of the crust. In the present section we consider that a generic mechanism is at work to produce a change in the number densities. In Section IV we shall evaluate the particle mass creation rate corresponding to the beta decay processes. Regarding the transfusion processes the corresponding creation rates are difficult to evaluate and we postpone their calculations to future work.

In any case, the three particle creation rates are not independent quantities, because charge conservation implies that

$$\Gamma_e = \Gamma_p, \quad (3)$$

whereas baryon number conservation leads to

$$\Gamma_p = -\Gamma_n, \quad (4)$$

meaning that only one creation rate is independent.

It is possible to simplify the treatment of the system considering that our analysis regards processes that happen at a time scale much larger than the electromagnetic time scale. Therefore, we can consider that electrons and protons are locked together to move with the same velocity [17], see however [18]. Moreover, charge neutrality implies that the number densities of electrons and protons are equal, *i.e.*  $n_e = n_p$ . Therefore, electrons and protons can be treated as a single charge-neutral fluid and henceforth we shall refer to this fluid as the “charged” component, employing the subscript  $c$  to label it. As a matter of fact, the system can be viewed as consisting of two fluids, with mass densities

$$\rho_n = m_n n_n \quad \text{and} \quad \rho_c = m_n n_c, \quad (5)$$

where  $m_n = m_p + m_e$  and  $n_e = n_p = n_c$ . The Euler equations obeyed by the two fluids are given by

$$(\partial_t + v_n^j \nabla_j)(v_i^n + \epsilon_n w_i) + \nabla_i(\tilde{\mu}_n + \Phi) + \epsilon_n w^j \nabla_i v_j^n = \frac{f_i^{\text{MF}}}{\rho_n}, \quad (6)$$

$$(\partial_t + v_c^j \nabla_j)(v_i^c - \epsilon_c w_i) + \nabla_i(\tilde{\mu}_c + \Phi) - \epsilon_c w^j \nabla_i v_j^c = -\frac{f_i^{\text{MF}}}{\rho_c} + (1 - \epsilon_n - \epsilon_c) \frac{\Gamma_n}{\rho_c} w_i, \quad (7)$$

where  $i, j$  label the space components, we have defined a chemical potential by mass  $\tilde{\mu}_x = \mu_x / m_n$ , and  $\mathbf{w} = \mathbf{v}_c - \mathbf{v}_n$  represents the relative velocity between the two fluids. The quantities  $\epsilon_n$  and  $\epsilon_c$  are the entrainment parameters, that are related to the fact that momenta and velocities of quasiparticles may not be aligned [19].

The gravitational potential,  $\Phi$ , obeys the Poisson equation

$$\nabla^2 \Phi = 4\pi G (\rho_n + \rho_c). \quad (8)$$

The last term on the right hand side of Eq. (7) is the rocket term. This force is due to the fact that when two fluids move with different velocities a change of one component into the other results in a variation of the momentum of each fluid component. This change in momentum can be viewed as a force proportional to the mass rate change,  $\Gamma_n$ , and to the relative velocity between the two fluids,  $\mathbf{w}$ . Actually, in Eqs. (6,7), one can see that the rocket term acts only on the charged component. The reason is that part of the rocket term contribution can be included in the

definition of the mutual friction force; here we have employed the same notation used in Ref. [15]. In the analysis of the possible dissipative mechanisms of star oscillations this term is usually neglected. Indeed, it is in general assumed that the neutron, proton and electron numbers are separately conserved quantities, that is  $\Gamma_p = \Gamma_e = \Gamma_n = 0$ .

The force term  $f_i^{\text{MF}}$  entering into both Euler equations corresponds to the mutual friction force between the two fluids. At the microscopic level it is due to the scattering of electrons off the magnetic field trapped in the superfluid vortices [17]. Indeed the core of the superfluid vortices is magnetized, because of the interaction between the neutron and the proton condensates. The mutual friction force can be expressed as

$$f_i^{\text{MF}} = 2\rho_n B' \epsilon_{ijk} \Omega_j w^k + 2\rho_n B \epsilon_{ijk} \hat{\Omega}^j \epsilon^{klm} \Omega_l w_m, \quad (9)$$

where the coefficients  $B, B'$  can be written as

$$B = \frac{\mathcal{R}}{1 + \mathcal{R}^2}, \quad \text{and} \quad B' = \frac{\mathcal{R}^2}{1 + \mathcal{R}^2}, \quad (10)$$

where  $\mathcal{R}$  is the dimensionless “drag” parameter [11]. The value of  $\mathcal{R}$  depends on the mechanism of trapping of the magnetic field inside superfluid vortices [17, 20]. The details of this mechanism are unknown and one can consider three different regimes: the weak drag regime,  $\mathcal{R} \ll 1$ , the strong drag regime  $\mathcal{R} \gg 1$  and the intermediate drag regime,  $\mathcal{R} \sim 1$ . For small values of the drag parameter one can express the coefficients  $B, B'$  as a function of the entrainment parameter. According with Ref. [11], one has that

$$B = 4 \times 10^{-4} \frac{\epsilon_c^2}{\sqrt{1 - \epsilon_c}} \left( \frac{x_c}{0.05} \right)^{7/6} \left( \frac{\rho}{10^{14} \text{g/cm}^3} \right)^{1/6} \quad \text{and} \quad B' \simeq B^2. \quad (11)$$

### III. PERTURBED HYDRODYNAMICAL EQUATIONS

A non-vanishing mass creation rate influences the evolution of the various hydrodynamical. Indeed, the continuity equations (1) as well as the Euler equations (7) depend on  $\Gamma_n$ . Therefore, in the analysis of the various modes of oscillations of a neutron star one has to take into account effects related to this term. In the present paper we only discuss its effect on the evolution of the r-modes of a superfluid neutron star, although it would be equally interesting to study its effect on other pulsation modes. In the following analysis of the hydrodynamical equations we also include the mutual friction force and we follow very closely the recent analysis of the r-mode oscillations developed in Ref. [21] for normal fluid stars, and extended to superfluid stars in Ref. [14].

As in Ref. [14], we study the linearized hydrodynamical equations for the perturbations around an equilibrium configuration of a neutron star rotating with constant angular velocity  $\Omega$ , the r-mode is purely axial to leading order in  $\Omega$  and we work in the equilibrium hydrodynamical quantities. and we assume that the background configuration is such that the two fluids move with the same velocity, thus at equilibrium  $\mathbf{w} = 0$ .

It is useful to write the Euler equations for the perturbed quantities using as degrees of freedom (dof) the center of mass displacement and the relative displacements between the neutron fluid and the charged fluid. We define the comoving velocity as

$$\delta \mathbf{v} = \frac{\rho_n}{\rho} \delta \mathbf{v}_n + \frac{\rho_c}{\rho} \delta \mathbf{v}_c, \quad (12)$$

and the countermoving, or relative, velocity as

$$\delta \mathbf{w} = \delta \mathbf{v}_c - \delta \mathbf{v}_n. \quad (13)$$

The continuity equation for the comoving degree of freedom is not affected by the rocket term and is given by

$$\partial_t \delta \rho + \nabla_j (\rho \delta v^j) = 0, \quad (14)$$

on the other hand the continuity equation for the countermoving dof depends on it. It is useful to employ as a second continuity equation the one for the charged fraction,  $x_c = \rho_c / \rho$ , which is given by

$$\partial_t \delta x_c = -\frac{1}{\rho} \nabla \cdot [x_c (1 - x_c) \rho \delta \mathbf{w}] - \delta \mathbf{v} \cdot \nabla x_c - \frac{\Gamma_n}{\rho}. \quad (15)$$

The rocket term affects the Euler equations for both the comoving and countermoving velocities and one has that

$$\partial_t \delta v_i + 2\epsilon_{ijk} \Omega^j \delta v^k + \frac{1}{\rho} \nabla_i \delta p - \frac{\delta \rho}{\rho^2} \nabla_i p + \nabla_i \delta \Phi = (1 - \bar{\epsilon}) \frac{\Gamma_n}{\rho} \delta w_i, \quad (16)$$

$$\partial_t (1 - \bar{\epsilon}) \delta w_i + \nabla_i (\delta \beta) + 2\bar{B}' \epsilon_{ijk} \Omega_j \delta w^k - 2\bar{B} \epsilon_{ijk} \hat{\Omega}^j \epsilon^{klm} \Omega_l \delta w_m = (1 - \bar{\epsilon}) \frac{\Gamma_n}{\rho} \delta w_i, \quad (17)$$

where here we have defined  $\bar{\epsilon} = \epsilon_c + \epsilon_n = \epsilon_n (1 + \rho_n / \rho_c) = \epsilon_n / x_c$ , and where

$$\delta \beta = \delta \tilde{\mu}_c - \delta \tilde{\mu}_n \quad (18)$$

and

$$\bar{B} = B / x_c, \quad \bar{B}' = 1 - B' / x_c. \quad (19)$$

The hydrodynamical equations can be studied employing a perturbative expansion of the various hydrodynamical variables in  $\Omega$ , the star rotation frequency. Actually, the expansion is in the parameter  $\Omega / \Omega_K$ , where  $\Omega_K$  is the Kepler frequency of the star. For superfluid systems, this expansion is particularly convenient, as one can show that the complicated system of equations for the comoving and countermoving degrees of freedom decouple as these variables depend on different powers of  $\Omega$ .

For our study we consider some simplifying, admittedly unrealistic, assumptions. We neglect the deformation of the star due to rotation, which affect the different hydrodynamical variables at order  $\Omega^2$ . We use the Cowling approximation, that is, we neglect perturbations of the gravitational potential associated with the oscillations of the star. As a further simplification, we also consider a model where the mass density of the star is uniform. As emphasized in the Introduction, our goal is to study the effect and impact of the rocket term in the evolution of the r-modes, and we leave for future studies a more realistic model of the star.

Oscillations of a fluid element of a stars can be described by the Lagrangian displacement vector  $\xi$ , which can be decomposed into a sum of toroidal and spheroidal components. Since neutron stars can be described employing the two fluid model, one defines comoving and countermoving displacements, respectively  $\xi_+$  and  $\xi_-$ , by means of the equations

$$\delta \mathbf{v} = \partial_t \xi_+ \propto \Omega \xi_+, \quad \delta \mathbf{w} = \partial_t \xi_- \propto \Omega \xi_-. \quad (20)$$

These two displacements describe the center of mass oscillation and the out of phase oscillation of the two fluids, respectively. We then expand these quantities in terms of toroidal and spheroidal components

$$\xi_+ = r \sum_{l,m} \left( 0, \frac{K_{lm}}{\sin \theta} \partial_\phi, -K_{lm} \partial_\theta \right) Y_{lm} + r \sum_{l,m} \left( S_{lm}, Z_{lm} \partial_\theta, \frac{Z_{lm}}{\sin \theta} \partial_\phi \right) Y_{lm}, \quad (21)$$

$$\xi_- = r \sum_{l,m} \left( 0, \frac{k_{lm}}{\sin \theta} \partial_\phi, -k_{lm} \partial_\theta \right) Y_{lm} + r \sum_{l,m} \left( s_{lm}, z_{lm} \partial_\theta, \frac{z_{lm}}{\sin \theta} \partial_\phi \right) Y_{lm}, \quad (22)$$

where  $Y_{lm}$  are the spherical harmonics. The fluctuations of the pressure and of the chemical potential difference can be written respectively as

$$\delta p = \rho g r \sum_{l,m} \zeta_{lm} Y_{lm}, \quad (23)$$

$$\delta \beta = g r \sum_{l,m} \tau_{lm} Y_{lm}, \quad (24)$$

where  $g = \Omega_0^2 r$  (with  $\Omega_0^2 = GM/R^3$ ) fixes the scale of pressure and chemical potential fluctuations. Notice that with these definitions,  $\tau_{lm}$  and  $\zeta_{lm}$  are dimensionless.

Since in a superfluid star one has two different kind of displacements, in principle one can have two different kind of r-mode oscillations, one associated with the comoving dof and one associated with the countermoving dof. Actually, the hydrodynamical variables defined above obey a complicated set of coupled differential equations, see [14], with couplings between comoving and countermoving displacements. However, as shown in [14], at the leading order in  $\Omega$ , one finds that the equation for the comoving displacement decouples and one can determine an analytic expression for the standard r-mode oscillation. Regarding the mode associated with the countermoving dof, it turns out to be a general inertial mode. That is, it is not a mode dominated by the toroidal components. However, for incompressible stars with uniform density one has that this inertial mode turns into an r-mode. We shall restrict to this case and analyze this r-mode oscillation in Sec. III B. We explicitly consider the effect of the mutual friction in the equations of motion, the reason is that in this way we can analyze the regime where the mutual friction coefficients,  $B$  and  $B'$ , are large. Therefore our results will explicitly depend on the values of these parameters.

### A. Standard r-mode oscillations

For the standard r-mode oscillations one assumes that the comoving toroidal displacement,  $K_{lm}$ , is of order unity, while the spheroidal comoving displacements are of order  $\Omega^2$ . All the counter-moving displacements turn out to be of order  $\Omega^2$  as well. Since the standard r-mode oscillation is dominated by  $K_{lm}$ , it is very similar to the r-mode oscillation in normal fluids [1], and can be easily determined after imposing proper boundary conditions [14]. To first order in the rotation frequency of the star, one has that the typical frequency of the oscillations (measured in the corotating frame) is

$$\omega_r = \frac{2m\Omega}{l(l+1)}. \quad (25)$$

In our analysis we restrict to analyze the case  $l = m = 2$ , which corresponds to the most unstable r-mode.

Regarding the pressure perturbations, they are of order  $\Omega^2$ , whereas  $\delta\beta \propto \Omega^4$  [12, 14]. The order in  $\Omega$  of the toroidal oscillations and of the pressure and chemical potential fluctuations are reported in the first line of Table I.

type of r-mode	$K_{lm}$	$k_{lm}$	$\zeta_{lm}$	$\tau_{lm}$	$S_{lm}, s_{lm}, Z_{lm}, z_{lm}$
standard r-mode	$\mathcal{O}(\Omega^0)$	$\mathcal{O}(\Omega^2)$	$\mathcal{O}(\Omega^2)$	$\mathcal{O}(\Omega^4)$	$\mathcal{O}(\Omega^2)$
superfluid r-mode	$\mathcal{O}(\Omega^2)$	$\mathcal{O}(\Omega^0)$	$\mathcal{O}(\Omega^4)$	$\mathcal{O}(\Omega^2)$	$\mathcal{O}(\Omega^2)$

TABLE I: Order in  $\Omega$  of the comoving and counter-moving displacements, of the pressure fluctuation and of the chemical potential fluctuation for the standard r-mode oscillation and for the superfluid r-mode oscillation.

For purposes of estimating the damping time scales associated to both mutual friction and the rocket term that we will carry out in Sec. V, we have to determine the solutions for the counter-moving dof. The equations governing the evolution of the various dynamical variables are reported in the Appendix A. Assuming constant mass density and hydrostatic equilibrium we find that  $\tau_{l+1}$  obeys the following differential equation

$$r^2 \tau_{l+1}'' = (A_1 + B_1 - 1)r\tau_{l+1}' + (A_2 B_2 - A_1 B_1)\tau_{l+1} - A_2 B_4 \frac{r^{l+1}}{R^2 - r^2}, \quad (26)$$

where the prime indicates differentiation with respect to  $r$  and the coefficients  $A_i, B_i$  are reported in Appendix A. As shown there, the last term on the right hand side of this equation arises because we have assumed that matter is in hydrostatic equilibrium. The differential equation has solution given by

$$\tau_{l+1}(r) = f(r) + C_1 r^{n_1} + C_2 r^{n_2}, \quad (27)$$

where  $f(r)$  is the particular solution of the differential equation and where  $C_1$  and  $C_2$  are the coefficients of the homogeneous solution, to be fixed by the boundary conditions. The exponents of the homogeneous solution are given by

$$n_{1,2} = \frac{A_1 + B_1 \pm \sqrt{(A_1 + B_1)^2 + 4(A_2 B_2 - A_1 B_1)}}{2}, \quad (28)$$

and it turns out that  $n_2$  is negative, meaning that in order to avoid divergences at  $r = 0$ , it must be  $C_2 = 0$ . It is interesting to note that for vanishing mutual friction one has that  $n_1 = l - 1$  and  $n_2 = -(l + 4)$ . As a second boundary condition we assume that the chemical potential difference vanishes at the surface of the star, that is  $\delta\beta(R) = 0$ .

For completeness we report the equation obeyed by the radial component of the counter-moving spheroidal displacement, which is given by

$$\xi_-^r = \frac{r^2 \tau_{l+1}'}{A_2} - \frac{A_1 r \tau_{l+1}}{A_2}. \quad (29)$$

### B. Superfluid r-mode oscillations

Assuming that  $k_{lm}$  is of order unity one finds that the spheroidal counter-moving displacements are of order  $\Omega^2$ . The driving force on the counter-moving displacement is the chemical potential difference which turns out to be of order  $\Omega^2$ . The order of the toroidal comoving displacement depends on the compressibility of the fluid. For a compressible



fluid it is of order  $\Omega^0$ , while for an incompressible fluid it is of order  $\Omega^2$ . The reason can be traced back to the fact that comoving oscillations are driven by pressure oscillations and it turns out that  $K_{lm} \sim \Omega^{-2}\zeta$ . For compressible fluids the pressure oscillations are proportional to chemical potential oscillations and therefore  $\zeta \sim \Omega^2$  and thus  $K_{lm}$  must be of order unity. Moreover, for this kind of mode, comoving spheroidal displacements turn out to be of the same order in  $\Omega$  of comoving toroidal displacements, meaning that for a compressible fluid this oscillation is a generic inertial mode and not an r-mode. Since for a compressible fluid various components of the displacements are of the same order in  $\Omega$ , one has to solve a system of coupled differential equations.

The situation is much easily tractable for incompressible fluids. In this case one can assume that spheroidal oscillations are of order  $\mathcal{O}(\Omega^2)$  and then toroidal oscillations turn out to be of the same order. The order in  $\Omega$  of the various displacements and of the pressure and chemical potential fluctuations for incompressible matter are reported in Table I. We shall restrict the analysis to the case of incompressible fluids, where the comoving and counter-moving dof decouple, with the superfluid r-mode oscillation dominated by the toroidal displacement  $k_{lm}$ . To first order in the rotation frequency of the star and to first order in the entrainment parameter, the typical frequency of the superfluid r-mode oscillation (measured in the corotating frame) is

$$\omega_r = \frac{2m\Omega}{l(l+1)}(1 + \bar{\epsilon}). \quad (30)$$

As for the standard r-mode, we restrict to analyze the case  $l = m = 2$ , which corresponds to the most unstable r-mode. Moreover we consider only small values of the entrainment.

The analysis of the the superfluid r-mode oscillation is very similar to the one we have performed for the standard r-mode oscillations, with the roles of  $K_{lm}$  and of the pressure oscillations interchanged with  $k_{lm}$  and the chemical potential oscillations. We find that for superfluid r-modes,  $k_{lm}$  obeys the same equation that  $K_{lm}$  obeys for standard r-modes, and the chemical potential fluctuation obeys the same equation that pressure fluctuation obey for standard r-modes. Regarding the pressure oscillation  $\zeta_{lm}$  one has to solve an equation analogous to Eq. (26), but without the last term on the right hand side, because we are now considering an incompressible fluid. We find that

$$\zeta_{l+1} = C_1 r^{s_1} + C_2 r^{s_2}, \quad (31)$$

where  $s_{1,2}$  depend on the parameters of the model. One of the two coefficients is always negative, and therefore in order to avoid the divergence at the origin, we have that

$$\zeta_{l+1} = C r^s. \quad (32)$$

We fix  $C$  by demanding that the comoving toroidal displacement,  $K_{lm}$ , is properly normalized, as in Ref. [13].

#### IV. URCA PROCESSES AND THE ROCKET TERM

In order to evaluate the damping timescale associated with the rocket term we need to evaluate the mass creation rate  $\Gamma_n$ . There are two main processes that are responsible of the change of the densities of neutrons, protons and electrons in neutron stars. The capture/release of neutrons from the ions in the crust and the Urca processes. The first process regards the properties of ions at extremely high densities and it is difficult to take into account properly. One should consider all the microscopic processes as well as macroscopic processes (like crust-quakes) that can lead to a *transfusion* of material between the crust and the underlying fluid. We are not aware of any calculation of the mass creation rate associated with the transfusion mechanisms. However, it is worth mentioning that such processes might become important for a self-consistent treatment of r-modes. R-mode oscillations have a radial component that can be of the order of 100 m. Such a radial displacement comes with a pressure oscillations that can lead to a change in composition of the crust and may lead to transfer of material between the crust and the underlying superfluid.

We postpone the treatment of the transfusion processes to future work. For the time being we restrict to Urca processes that take place among protons, neutrons and electrons. It was found in Ref.[22], that for certain realistic equations of state the direct Urca processes are allowed when the star density exceeds the nuclear saturation density  $\rho_0 = 2.8 \times 10^{14} \text{ g cm}^{-3}$ , and the proton fraction exceeds the threshold value  $x_p^c = \frac{1}{9}$ . We shall restrict to consider such processes assuming that these conditions are realized in the interior of neutron stars. Henceforth we shall take  $x_p = x_p^c$ , but our results do not strongly depend on the precise value of  $x_p$  as far as  $x_p \geq x_p^c$ . The direct Urca reactions are given by

$$n \rightarrow p + e^- + \bar{\nu}_e, \quad p + e^- \rightarrow n + \nu_e, \quad (33)$$

and the corresponding reaction rates are given by

$$\Gamma_{\text{Urca}}^d = \int \prod_{i=n,p,e,\nu} \left[ \frac{d^3 p_i}{(2\pi)^3 2E_i} \right] f_n (1 - f_p) (1 - f_e) \sum_{\text{spins}} |M|^2 (2\pi)^4 \delta^{(4)}(P_i - P_f), \quad (34)$$

for the neutron decay process, and by

$$\Gamma_{\text{Urca}}^c = \int \prod_{i=n,p,e,\nu} \left[ \frac{d^3 p_i}{(2\pi)^3 2E_i} \right] f_p f_e (1 - f_n) \sum_{\text{spins}} |M|^2 (2\pi)^4 \delta^{(4)}(P_i - P_f), \quad (35)$$

for the electron capture process. In both reaction rates,  $|M|^2$  is the squared of the scattering cross section for the weak interaction process (see Refs. [22] and [23] for its explicit value), and

$$f_i = \frac{1}{e^{(E_i - \mu_i)/T} - 1}, \quad (36)$$

is the Fermi-Dirac distribution function of the particle specie  $i = n, p, e$ .

In equilibrium, or quasi-equilibrium, the neutron decay and the electron capture processes balance, meaning that  $\bar{\Gamma}_{\text{Urca}}^d = \bar{\Gamma}_{\text{Urca}}^c$ . In such a case, one has  $\Gamma_n = \Gamma_p = 0$ , and the rocket terms in the hydrodynamical equations (16) and (17) vanish. At equilibrium the velocities of the proton and neutron fluids should be the same, and further, the chemical potentials should fulfill the relation

$$\bar{\mu}_n = \bar{\mu}_c = \bar{\mu}_p + \bar{\mu}_e. \quad (37)$$

Therefore, a perturbation around the equilibrium distribution is related to a difference in the velocities of the two fluids and/or a deviation from chemical equilibrium. Out of equilibrium the neutron decay process and the electron capture process do not compensate, resulting in a net source (or sink) of neutrons, and thus, in a non-vanishing value of  $\Gamma_n$ , given by

$$\Gamma_n = \Gamma_{\text{Urca}}^d - \Gamma_{\text{Urca}}^c. \quad (38)$$

In order to evaluate this quantity we consider small perturbations of the equilibrium distribution functions,  $f_i = \bar{f}_i + \delta f$ ; upon substituting the perturbed distribution function in Eqs.(34,35) and considering terms linear in the perturbation, we obtain the mass creation rate

$$\Gamma_n = -\frac{1}{T} \left( \delta\mu_n - \delta\mu_c + \frac{m_n}{2} (1 - \epsilon_n - \epsilon_p) (\delta\mathbf{w})^2 \right) \bar{\Gamma}_{\text{Urca}}, \quad (39)$$

in agreement with the results of Ref. [15]. Note that the last term in Eq. (39) gives the correction to the chemical equilibrium due to the relative motion between the two fluids.

The value of  $\bar{\Gamma}_{\text{Urca}}$  depends on the particular neutron superfluid phase that is realized in the star. In Ref. [23] three phases have been considered, one corresponding to a  $^1S_0$  condensate and two corresponding to the  $^3P_2$  condensate with  $m_j = 0$  and  $m_j = 2$ . In any superfluid phase the Urca reaction rate is suppressed with respect to the corresponding value in unpaired matter. The reason is that the phase space available for scattering is reduced in the superfluid phase. This leads to a suppression of the neutrino emission rate and of the bulk viscosity coefficient in the cores of superfluid neutron stars [23]. In our case this leads to a suppression of the rocket term force. However, it is worth remarking that while the suppression is exponentially large for the  $^1S_0$  and for the  $^3P_2$  condensate with  $m_j = 0$  (of order  $\exp(-\Delta/T)$ , with  $\Delta$  the corresponding energy gap), it is much smaller for the  $^3P_2$  condensate with  $m_j = 2$ . This is due to the fact that in the latter case the quasiparticle fermionic dispersion law is gapless. We report the results of Ref. [23] regarding these three superfluid phases in Appendix B.

In the situation where the direct Urca processes are not allowed, one should consider modified Urca processes,

$$n + N \rightarrow p + N + e^- + \bar{\nu}_e, \quad p + N + e^- \rightarrow n + N + \nu_e, \quad (40)$$

where  $N$  is an additional nucleon. However, the reaction rates of the modified Urca processes are 3-5 orders of magnitude lower than those of direct Urca processes [24], and the effect of the rocket term in the damping of r-modes will be weaker. For the time being, we will assume that the conditions for the direct Urca processes are allowed in the star.

In general, one can express the mass creation rate as

$$\Gamma_n = \frac{\delta\beta^w}{k_B T} \Xi, \quad (41)$$



where  $k_B$  is the Boltzmann constant and we have defined

$$\delta\beta^w = m_n \delta\beta - \frac{m_n}{2}(1 - \epsilon_n - \epsilon_c)(\delta\mathbf{w})^2, \quad (42)$$

where in the case when the Urca processes are allowed  $\Xi = \bar{\Gamma}_{\text{Urca}}$  and its value depends on the superfluid phase considered.

## V. STABILITY ANALYSIS

Given the mass creation rate, we can now evaluate the damping timescale associated with the rocket term and compare the result with the timescale of different processes. In principle one can determine the damping times associated with any mechanism from the imaginary part of the mode frequency. However, in the present analysis we shall limit ourselves to the energy integral estimates.

First we evaluate the energy associated with any oscillation of the star. In the frame corotating with the star, see [13] for more details, it can be written as

$$E = \frac{1}{2} \int \left( \rho (|\delta\mathbf{v}|^2 + (1 - \bar{\epsilon})x_p(1 - x_p)|\delta\mathbf{w}|^2) + \sum_{i,j=n,c} \mathcal{P}_{ij} \delta\rho_i \delta\rho_j^* - \frac{|\nabla\delta\Phi|^2}{4\pi G} \right) dV, \quad (43)$$

where the integral is extended to the volume of the superfluid core of the star and we have defined  $\mathcal{P}_{ij} = \frac{\partial\mu_i}{\partial\rho_j}$ . The energy stored in the oscillations contains a kinetic energy contribution,  $E_k$ , and a potential energy contribution,  $E_{\text{pot}}$ .

The variation of the total energy due to mutual friction and rocket term dissipations can be obtained using the continuity and Euler equations for the oscillations, integrating by parts, and discarding surface terms. In this way we have that the total energy variation associated with the oscillations is given by

$$\frac{\partial E}{\partial t} = \frac{\partial(E_k + E_{\text{pot}})}{\partial t} = - \int dV \left( |\delta\mathbf{w} \cdot \mathbf{f}_{\text{MF}}| + |\delta\beta^w|^2 \frac{\Xi}{T} \right). \quad (44)$$

These results are in agreement with the general considerations for the entropy generation due to both mutual friction and the rocket term of Ref. [15]. Further, the requirements of positive entropy production allow us to guarantee that these terms always represent energy losses, and not gains.

The damping timescale associated to the mutual friction is then given by

$$\frac{1}{\tau_{\text{MF}}} = -\frac{1}{2E} \left( \frac{\partial E}{\partial t} \right)_{\text{MF}} = \frac{1}{2E} \int dV |\delta\mathbf{w} \cdot \mathbf{f}_{\text{MF}}|, \quad (45)$$

while for the rocket term one has that

$$\frac{1}{\tau_{\text{RT}}} = -\frac{1}{2E} \left( \frac{\partial E}{\partial t} \right)_{\text{RT}} = \frac{1}{2E} \int dV |\delta\beta^w|^2 \frac{\Xi}{T}. \quad (46)$$

The timescale associated with gravitational wave emission is instead given by

$$\frac{1}{\tau_{\text{gw}}} = -\frac{1}{2E} \left( \frac{\partial E}{\partial t} \right)_{\text{gw}} = \frac{1}{2E} \omega_r \sum_l N_l \omega_i^{2l+1} (|D_{lm}|^2 + |J_{lm}|^2), \quad (47)$$

where  $\omega_r$  is the r-mode frequency and  $\omega_i = \omega_r - m\Omega$  is the frequency measured by an inertial observer. Moreover

$$N_l = \frac{4\pi G}{c^{2l+1}} \frac{(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2}, \quad (48)$$

and  $D_{lm}$  and  $J_{lm}$  are the mass and current multipoles respectively.

The three timescales  $\tau_{\text{MF}}$ ,  $\tau_{\text{RT}}$  and  $\tau_{\text{gw}}$  depend on the particular r-mode oscillation considered. In the following sections we shall separately analyze standard r-modes and superfluid r-modes. In order to make our results comparable with the analysis of Ref. [12] we define the entrainment parameter

$$\epsilon = \frac{\epsilon_c x_c}{1 - x_c - \epsilon_c}, \quad (49)$$

and according with [12] we consider the range of values  $\epsilon \leq 0.06$ .

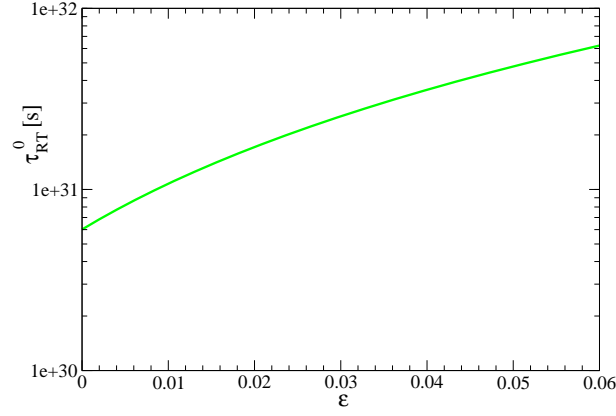


FIG. 1: Damping time  $\tau_{RT}^0$  associated with the rocket term as a function of the entrainment parameter  $\epsilon$  (defined in Eq.(49)) for the weak drag regime. We have taken  $T = 10^9$  K and a critical superfluid temperature  $T_c = 10^{10}$  K. The star has uniform density  $\rho = 2.5\rho_0$ , with radius  $R = 10$  km and mass  $M \simeq 1.47M_\odot$ .

### A. Timescales for standard r-modes

The standard r-mode oscillation is predominantly associated with toroidal comoving displacements. According with Table I and equations (20) we have that  $\delta v \propto \Omega$ , while  $\delta w \propto \Omega^3$  to leading order in the  $\Omega$  expansion. Thus the leading term in the total energy, Eq. (43), is associated with the comoving oscillations and therefore  $E \propto \Omega^2$ .

We can now evaluate the power counting in  $\Omega$  of the various timescales. For the mutual friction timescale, we have from Eq.(9) that  $f_{MF} \sim \Omega^4$  and it follows from Eq.(45) that  $\tau_{MF} \propto \Omega^{-5}$ . As for the damping time of the rocket term in Eq.(46), we have that  $\tau_{RT} \propto \Omega^{-6}$ . This comes from the fact that  $\delta\beta \propto \Omega^4$  at the leading order in  $\Omega$ . This also means that in Eq. (42) one can neglect the second term on the right hand side, because  $(\delta\mathbf{w})^2 \propto \Omega^6$ . The timescale associated with gravitational-wave emission for standard r-modes has been computed in Refs. [1, 2, 5, 6, 25, 26]. For a star of constant density, the growth timescale of the  $l = 2$  current multipole is given by

$$\tau_{gw} \simeq 22 \left( \frac{M}{1.4M_\odot} \right)^{-1} \left( \frac{R}{10\text{km}} \right)^{-4} \left( \frac{P}{1\text{ms}} \right)^6 \text{ s}, \quad (50)$$

and therefore  $\tau_{gw} \propto \Omega^{-6}$ .

In r-mode analysis it is useful to introduce the frequency independent quantities  $\tau_{MF}^0$ ,  $\tau_{RT}^0$ ,  $\tau_{gw}^0$  which are obtained from the corresponding timescales extracting the  $\Omega$  dependence. For standard r-modes we have that

$$\frac{1}{\tau_{MF}} = \frac{1}{\tau_{MF}^0} \left( \frac{\Omega}{\sqrt{\pi G \rho}} \right)^5, \quad \frac{1}{\tau_{RT}} = \frac{1}{\tau_{RT}^0} \left( \frac{\Omega}{\sqrt{\pi G \rho}} \right)^6, \quad \frac{1}{\tau_{gw}} = \frac{1}{\tau_{gw}^0} \left( \frac{\Omega}{\sqrt{\pi G \rho}} \right)^6. \quad (51)$$

Since we have explicitly considered the mutual friction force in the Euler equations, the mutual friction timescale,  $\tau_{MF}^0$ , and the rocket term timescale,  $\tau_{RT}^0$ , depend on the drag parameter. In agreement with Ref. [14] we find that for  $R \sim 1$  the mutual friction is able to damp the r-mode oscillations, while for large or small values of  $R$  the damping time is too long.

Regarding the rocket term damping timescale, we compute the numerical value of the coefficient  $\Xi$  in Eq. (41). In Fig. 1 we report the plot of the rocket term damping timescale versus entrainment. We consider the weak drag regime,  $\mathcal{R} \ll 1$ , and that  $nn$  pairing takes place in the  ${}^3P_2(m_J = 2)$  channel, which has the smallest reduction factor (reported in Appendix B, and corresponding to case C). However, results with different values of  $\mathcal{R}$  and with pairing in the  ${}^1S_0$  channel and in the  ${}^3P_2(m_J = 0)$  lead to very similar results. We assume throughout that the critical temperature of the superfluid phase is  $T_c = 10^{10}$  K. We find that  $\tau_{RT}^0$  does weakly depend on the entrainment parameter. The damping is stronger for small values of the entrainment, but in any case the damping time is extremely large.

More in detail, the rocket term damps the r-mode instability for

$$\tau_{RT} \leq \tau_{gw}, \quad (52)$$

and considering that the  $\tau_{RT}$  has the same frequency dependence of  $\tau_{gw}$ , it follows that the condition for damping is independent of the rotation frequency of the star and can be written as

$$\tau_{RT}^0 \leq \tau_{gw}^0. \quad (53)$$

Thus, when this condition is realized, the star is rotationally stable for any value of  $\Omega$ , as far as the  $\Omega/\Omega_K$  expansion makes sense.

However,  $\tau_{\text{gw}}^0$  is of order of tens of second, and as is clear from Fig. 1, the timescale of the rocket term is way too big for damping the r-mode instability. More precisely, considering the temperature dependence of the rocket term, we find that for temperatures below  $10^{15}\text{K}$  the damping timescale of the rocket term is larger than the growth timescale of gravitational waves. Therefore the effect of the rocket term is completely negligible in the analysis of standard r-mode instabilities. The qualitative reason why the rocket term effect is negligible is related to the fact that it is associated with countermoving displacements, which are subleading in standard r-mode oscillations. The suppression in  $\Omega$  can only be compensated by a very efficient particle conversion mechanism. In the present analysis we have assumed that particle conversion is due to direct Urca process, which is the most efficient microscopic decay one can consider. Therefore, the rocket term can have a significant effect on standard r-mode oscillations only if a very efficient macroscopic (transfusion) mechanism of particle conversion is at work in the neutron star. weak decays.

### B. Timescales for superfluid r-modes

The superfluid r-mode oscillation for incompressible superfluids is dominated by toroidal countermoving displacements. According with Table I, we have that  $\delta w \propto \Omega$  and  $\delta v \propto \Omega^3$ , to leading order in the  $\Omega$  expansion. The total energy in Eq. (43) is now dominated by the countermoving displacement and the power counting in  $\Omega$  is the same one has for standard r-modes, *i.e.*  $E \propto \Omega^2$ .

Regarding the mutual friction timescale, we have from Eq.(9) that  $f_{\text{MF}} \sim \Omega^2$ . Therefore in the case of superfluid r-modes the mutual friction force is much larger than in standard r-modes. This is clearly due to the fact that mutual friction force is proportional to the countermoving displacement, which are dominant for superfluid r-modes. From Eq.(45) we have that  $\tau_{\text{MF}} \propto \Omega^{-1}$ , which is four orders in  $\Omega$  smaller than the corresponding expression for standard r-modes.

As for the damping time of the rocket term, we now have that  $\delta\beta \propto \Omega^2$  and therefore from Eq.(46) we have that  $\tau_{\text{RT}} \propto \Omega^{-2}$ . Also in this case the timescale is much shorter than for standard r-modes, and for the same reason.

The timescale associated with gravitational-wave emission for superfluid r-modes is much larger than the corresponding value for standard r-modes. The reason is that gravitational wave emission is only associated with comoving displacement which are suppressed with respect to the countermoving modes. As a matter of fact the mass and current multipoles turn out to be both of order  $\Omega^3$  and we find that  $\tau_{\text{gw}} \propto \Omega^{-10}$ . Thus, for superfluid r-modes we have that

$$\frac{1}{\tau_{\text{MF}}} = \frac{1}{\tau_{\text{MF}}^0} \left( \frac{\Omega}{\sqrt{\pi G \rho}} \right), \quad \frac{1}{\tau_{\text{RT}}} = \frac{1}{\tau_{\text{RT}}^0} \left( \frac{\Omega}{\sqrt{\pi G \rho}} \right)^2, \quad \frac{1}{\tau_{\text{gw}}} = \frac{1}{\tau_{\text{gw}}^0} \left( \frac{\Omega}{\sqrt{\pi G \rho}} \right)^{10}. \quad (54)$$

We shall now analyze in detail the stability of the superfluid r-modes. In addition to mutual friction and rocket term one has to consider shear and bulk viscosity as well. We have evaluated both of these quantities for superfluid r-modes and found that

$$\frac{1}{\tau_{\text{sv}}} = \frac{1}{\tau_{\text{sv}}^0} \left( \frac{\Omega}{\sqrt{\pi G \rho}} \right), \quad \frac{1}{\tau_{\text{bv}}} = \frac{1}{\tau_{\text{bv}}^0} \left( \frac{\Omega}{\sqrt{\pi G \rho}} \right)^2. \quad (55)$$

In the numerical evaluation of the bulk viscosity timescale we have employed the results of [23], where the bulk viscosity is evaluated for direct Urca processes, but taking into account the reduction factor associated with the presence of the superfluid phase.

The study of the stability of r-mode oscillations now requires to take into account all the different timescales simultaneously. The relation for the stability is given by

$$-\frac{1}{\tau_{\text{gw}}} + \frac{1}{\tau_{\text{sv}}} + \frac{1}{\tau_{\text{bv}}} + \frac{1}{\tau_{\text{MF}}} + \frac{1}{\tau_{\text{RT}}} = 0, \quad (56)$$

meaning that when this relation is satisfied one has a critical condition for stability. When the quantity on the left hand side is negative, then the mode is unstable.

In Fig. 2 we report the result for the superfluid r-mode “instability window” for a star with uniform density,  $\rho = 2.5\rho_0$ , and  $R = 10$  km. The dashed red line represents the instability window in the absence of the rocket term and mutual friction. The region above the dashed red line is unstable with respect to gravitational wave emission when only shear and bulk viscosity damping mechanisms are considered. At low temperature, shear viscosity is the dominant dissipative mechanism. With increasing temperature shear viscosity is less efficient and it can damp r-mode

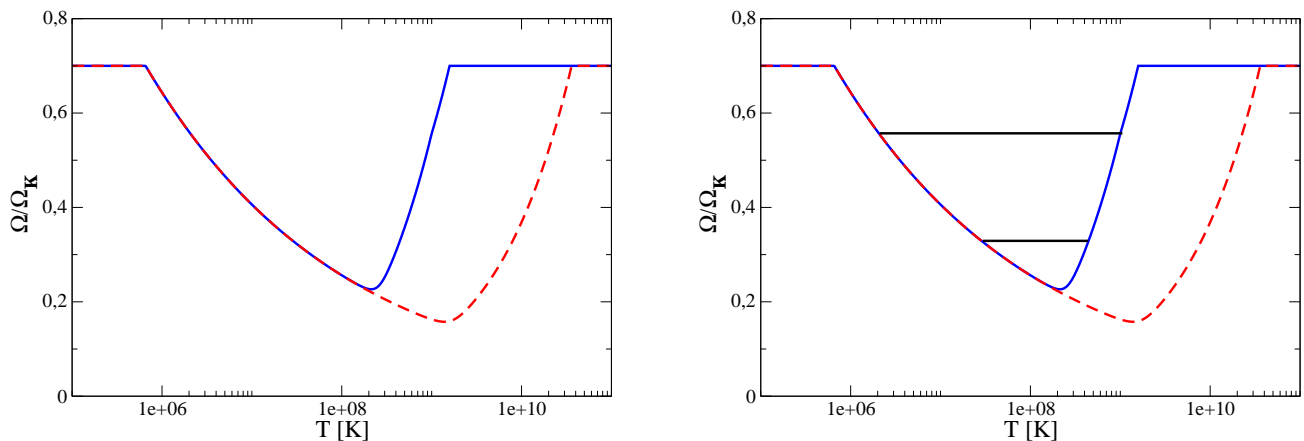


FIG. 2: Instability window of the superfluid r-modes of a star with uniform density  $\rho = 2.5\rho_0$ , with radius  $R = 10$  km and mass  $M \simeq 1.47M_\odot$ . In both panels, the dashed red line represents the instability window in the absence of the rocket term. The full blue line represents the instability window with the inclusion of the rocket term. The left panel corresponds to the case of vanishing entrainment and therefore vanishing mutual friction. In the right panel we consider two values of entrainment. The horizontal full lines correspond to the effect of the mutual friction for  $\epsilon = 0.0002$ , lower line, and  $\epsilon = 0.002$  upper line. In our simplified model of star, the mutual friction is independent of the temperature.

oscillations for smaller and smaller values of the frequency. The behavior of bulk viscosity is the opposite and starts to damp r-mode oscillations for temperatures of the order of  $10^{10}$  K. The full blue line represents the effect of the rocket term. It has a behavior qualitatively similar to the bulk viscosity, but it becomes effective at smaller temperatures. Therefore, the instability window with the inclusion of the rocket term is much reduced. In the left panel of Fig. 2 we have assumed zero entrainment and hence, vanishing mutual friction. In the right hand panel of Fig. 2 we include the effect of mutual friction. In the simplified model of neutron star that we are considering, mutual friction is independent of the temperature. We consider only the case of small drag parameter, where  $B$  and  $B'$  are given in Eq.(11) and are functions of the entrainment. The lower horizontal line corresponds to  $\epsilon = 0.0002$ , while the upper line corresponds to  $\epsilon = 0.002$ . In agreement with [13], we find that for values of  $\epsilon \sim 0.02$  the effect of the mutual friction is to damp superfluid r-modes for any value of the frequency.

## VI. CONCLUSION

Superfluid neutron stars are characterized by various oscillation modes. Of particular interest are r-mode oscillations, because they are unstable via gravitational wave emission, meaning that they would lead to the spin-down of the star, unless some dissipative mechanism sets in and damps these oscillations.

We have considered a very simple model of star comprising a fluid of neutrons, protons and electron, with uniform density and no crust. We have assumed that protons and electrons are locked together by the electromagnetic interaction and therefore the system can be described as made by two fluids: the neutron superfluid and the charged component.

In this model one has to consider two different r-mode oscillations. The standard r-mode oscillation, which are predominantly toroidal comoving displacements of the two fluids, and the superfluid r-mode oscillations which are associated with toroidal countermoving displacements. In realistic neutron stars these two modes are coupled, however, in the limit of small rotation frequency and assuming that the star has a uniform density and is incompressible, they decouple.

We have evaluated the effect of the rocket term on the stability of these two r-mode oscillations. In our analysis we have included the effect of mutual friction and of viscosities and compared the corresponding timescales with the timescale of gravitational wave emission. We find that the dissipative mechanism associated with the rocket term is not efficient in damping standard r-mode oscillations. The reason is that standard r-modes are predominantly comoving modes, while the rocket term is associated with the countermoving flow, which is subleading. On the other hand, this mechanism becomes efficient in damping the superfluid r-mode instability. The reason is that in this case the countermoving mode is the leading oscillation of the system and we find that for temperatures of the order of  $10^9$  K the rocket term has a typical timescale comparable with the timescale of gravitational wave emission. Quite interestingly, for superfluid r-modes the rocket term becomes an efficient damping mechanism before bulk

viscosity sets in. In agreement with [13] we find that the superfluid r-mode oscillations can be damped as well by the mutual friction force for sufficiently large values of the entrainment parameter. Unlike the mutual friction force, the rocket term is not strongly dependent on the entrainment parameter and it reduces the instability window even for vanishing entrainment. Since the entrainment parameter is a poorly known coefficient, the rocket term represents an interesting mechanism for superfluid r-mode damping.

It would be interesting to extend the present analysis to more realistic equation of states, in order to see whether the effect persists. It would be also quite interesting to study the effect of “transfusion” of neutrons and protons from the crust to the outer core. For standard r-modes we find that the damping timescale of the rocket term is quite large, however this timescale would be reduced if the transfusion of nucleons from the crust to the core results in a mass rate larger than the one due to weak processes. In this case the rocket term might be able to reduce the instability window for standard r-modes as well.

We have limited our analysis to the region of small chemical potential fluctuations,  $\delta\beta < T$ . As in Ref. [27], it would be interesting to extend the present analysis to the suprathreshold region,  $\delta\beta > T$ , in order to study the spind-down time of unstable stars. The rocket term might also be relevant for hybrid stars with a quark matter core. In that case one should consider weak processes between quarks of different flavors and transfusion processes between nuclear matter and quark matter.

### Appendix A: Evolution equations

Assuming uniform density the comoving and countermoving modes decouple [14]. For a star with uniform density one can impose hydrostatic equilibrium obtaining

$$P(r) = G \frac{2\pi}{3} (R^2 - r^2) \rho^2, \quad (\text{A1})$$

where we have assumed that the pressure vanishes at the surface of the star. Here  $R = 10$  km is the radius of the star and we shall consider a mass density  $\rho = 2.5\rho_0$ , where  $\rho_0$  is the saturation density of nuclear matter. With these values we obtain that the mass of the star is  $M \simeq 1.47M_\odot$ , where  $M_\odot$  is the mass of the sun. In this case we have that the pressure and the various components of the countermoving mode obey the following set of equations [14]

$$k_{lm} = a s_{l+1} + b z_{l+1} \quad (\text{A2})$$

$$\tau_{l+1} = c k_{lm} + d z_{l+1} + e s_{l+1} \quad (\text{A3})$$

$$r \frac{\partial \tau_{l+1}}{\partial r} = -2\tau_{l+1} - f k_{lm} - g z_{l+1} + h s_{l+1} \quad (\text{A4})$$

$$r \frac{\partial s_{l+1}}{\partial r} = -3s_{l+1} - \frac{V}{\Gamma} \frac{1}{1 - x_p} \zeta_{l+1} + p z_{l+1}, \quad (\text{A5})$$

where the various coefficients have been derived in Ref. [14] and for  $l = m$  are given by

$$a = \frac{\bar{B} - i\bar{B}'}{(1 - \bar{\epsilon}) - \bar{B}' - i\bar{B}} \frac{1}{\sqrt{2l+3}} \quad (\text{A6})$$

$$b = (l+2)a \quad (\text{A7})$$

$$c = l\omega^2(\bar{B} - i\bar{B}') \frac{1}{\sqrt{2l+3}} \quad (\text{A8})$$

$$d = \frac{\omega^2}{l+2} \left( (l+2)(1 - \bar{\epsilon}) - l\bar{B}' - i\bar{B} \frac{2+11l+8l+2l^2}{5+2l} \right) \quad (\text{A9})$$

$$e = -\frac{\omega^2}{l+2} \left( l\bar{B}' - i\bar{B} \frac{2-l}{5+2l} \right) \quad (\text{A10})$$

$$f = -(l+1)c \quad (\text{A11})$$

$$g = -p e \quad (\text{A12})$$

$$h = \omega^2 \left( (1 - \bar{\epsilon}) - 2i\bar{B} \frac{(l+1)^2}{5+2l} \right) \quad (\text{A13})$$

$$p = (l+1)(l+2), \quad (\text{A14})$$

where  $\tilde{\omega} = \Omega/\Omega_0$  and  $\omega = \sigma/\Omega_0$ , with  $\Omega_0 = \sqrt{GM/R^3}$ . In Eq. (A5) we have that

$$V = \frac{gr\rho}{P} \quad \text{and} \quad \Gamma = \frac{d \log P}{d \log \rho}, \quad (\text{A15})$$

which depend on the equation of state. Since we assume hydrostatic equilibrium we have from Eq. (A1) that

$$V = \frac{2r^2}{R^2 - r^2} \quad \text{and} \quad \Gamma = 2. \quad (\text{A16})$$

Finally, according with Ref. [14], we have that the pressure fluctuations are given by

$$\zeta_{l+1} = 2 \frac{\omega \tilde{\omega}}{\sqrt{2l+3}} \frac{l}{l+1} \frac{r^{l-1}}{R^{l-1}}. \quad (\text{A17})$$

Upon substituting the expressions above in the equations (A2),(A3),(A4) and (A5) and expressing  $k_{lm}$  and  $z_{l+1}$  in terms of  $s_{l+1}$  and  $\tau_{l+1}$  we have two coupled differential equations for  $s_{l+1}$  and  $\tau_{l+1}$ . These equations can be written as a second order differential equation

$$r^2 \tau_{l+1}'' = (A_1 + B_1 - 1) r \tau_{l+1}' + (A_2 B_2 - A_1 B_1) \tau_{l+1} - A_2 B_4 \frac{r^{l+1}}{R^2 - r^2}, \quad (\text{A18})$$

where

$$A_1 = -2 - \frac{fb + g}{cb + d}, \quad (\text{A19})$$

$$A_2 = -f \frac{ad - be}{cb + d} + g \frac{ac + e}{cb + d} + h, \quad (\text{A20})$$

$$B_1 = -3 - p \frac{ac + e}{cb + d}, \quad (\text{A21})$$

$$B_2 = \frac{p}{cb + d}, \quad (\text{A22})$$

$$B_4 = \frac{z}{1 - x_p}. \quad (\text{A23})$$

The analysis of the superfluid r-modes is analogous to the one we have done for the standard r-modes. However, in order to have an r-mode oscillation and not a generic inertial mode, one has to assume that the fluid is incompressible [14]. In this case  $\Gamma \rightarrow \infty$  and the differential equations one has to solve are simpler.

## Appendix B: Reaction rates for the Urca processes in different superfluid phases

The reaction rates for the direct Urca processes that may occur in superfluid neutron stars depend on the particular  $pp$  and  $nn$  superfluid condensates that can be realized. Various superfluid phases have been considered in Ref. [23], and the corresponding reaction rates have been evaluated; here we review some of their results.

If there is no superfluidity, the reaction rate for the direct Urca process can be expressed as

$$\bar{\Gamma}_{\text{Urca}} = (\Delta I) 1.667 \times 10^{32} (1 - \epsilon_c) \left( 1 - \epsilon_c \frac{x_c}{1 - x_c} \right) \left( \frac{\rho_c}{\rho_0} \right)^{1/3} T_9^5 \Theta_{npe} \text{ cm}^{-3} \text{ s}^{-1}, \quad (\text{B1})$$

where  $T_9$  is the temperature in units of  $10^9$  K. The step function  $\Theta_{npe}$  is 1 if the direct Urca process is allowed (see Ref. [22]), it is 0 otherwise. The factor  $(\Delta I)$  is a statistical factor that depends on the phase of nuclear matter under consideration. When all particles are in the normal phase, one finds that

$$\Delta I = \Delta I_0 = \frac{17\pi^4}{60}. \quad (\text{B2})$$

When neutrons or protons are superfluid this quantity is multiplied by a reduction factor,  $R_X$ , and one has that

$$\Delta I = \Delta I_0 R_X. \quad (\text{B3})$$

Here we consider only  $nn$  pairing, and the index  $X = A, B, C$ , depending on whether neutrons pair in the  $^1S_0$  channel (which corresponds to  $X = A$ ), in the  $^3P_2(m_J = 0)$  channel (which corresponds to  $X = B$ ), or in the  $^3P_2(m_J = 2)$



channel (which corresponds to  $X = C$ ). In Ref. [23] it is found that for temperatures smaller than the critical temperature of the corresponding superfluid phase, the reduction factors are well approximated by

$$R_A = \left(0.2787 + \sqrt{(0.7213)^2 + (0.1564v_A)^2}\right)^{3.5} \exp\left(2.9965 - \sqrt{(2.9965)^2 + v_A^2}\right), \quad (\text{B4})$$

$$R_B = \left(0.2854 + \sqrt{(0.7146)^2 + (0.1418v_B)^2}\right)^3 \exp\left(2.0350 - \sqrt{(2.0350)^2 + v_B^2}\right), \quad (\text{B5})$$

$$R_C = \frac{0.5 + (0.1086v_C)^2}{1 + (0.2347v_C)^2 + (0.2023v_C)^4} + 0.5 \exp\left(1 - \sqrt{1 + (0.5v_C)^2}\right), \quad (\text{B6})$$

where  $v_X = \frac{\Delta_X(T)}{k_B T}$ , and  $\Delta_X(T)$  is the temperature dependent superfluid gap of the phase under consideration. For temperatures larger than the critical temperature the reduction factors are equal to one. It is worth noticing that while in the phase *A* and *B* the reaction rates are suppressed exponentially at low  $T$ , the reduction factor for the phase *C* varies as  $T^2$ . The case when both neutrons and protons are in a superfluid phase is also considered in Ref. [23], where the reduction factors are also numerically computed.

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